

Half-plane capacity and conformal radius

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January 30, 2012

Abstract

In this note, we show that the half-plane capacity of a subset of the upper half-plane is comparable to a simple geometric quantity, namely the euclidean area of the hyperbolic neighborhood of radius one of this set. This is achieved by proving a similar estimate for the conformal radius of a subdomain of the unit disc, and by establishing a simple relation between these two quantities.

1 Introduction and results

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ be the upper half-plane. A bounded subset $A \subset \mathbb{H}$ is called a *hull* if $\mathbb{H} \setminus A$ is a simply connected region. The *half-plane capacity* of a hull A is the quantity

$$\text{hcap}(A) := \lim_{z \rightarrow \infty} z [g_A(z) - z],$$

where $g_A : \mathbb{H} \setminus A \rightarrow \mathbb{H}$ is the unique conformal map satisfying the hydrodynamic normalization $g(z) = z + O(\frac{1}{z})$ as $z \rightarrow \infty$. It appears frequently in connection with the Schramm-Loewner Evolution SLE, since it serves as the conformally natural parameter in the chordal Loewner equation, see [1]. In the study of SLE, one often needs estimates of $\text{hcap}(A)$ in terms of geometric properties of A . The definition of hcap in terms of conformal maps (or in terms of Brownian motion as in [1]) does not immediately yield such estimates. The purpose of this note is to provide a geometric quantity that is comparable to $\text{hcap}(A)$, via a simple relation between half-plane capacity and conformal radius.

Theorem 1.1. *The half-plane capacity and the (euclidean) area of the hyperbolic neighborhood of radius one are comparable,*

$$\text{hcap}(A) \asymp |N(A)|.$$

*Research supported by NSF Grant DMS-0800968.

More precisely, there are absolute constants $C_1, C_2 > 0$ so that

$$C_1 |N(A)| \leq \text{hcap}(A) \leq C_2 |N(A)|,$$

where $|N(A)|$ denotes the (euclidean) area of the hyperbolic neighborhood of radius one of A and

$$N(A) = \{z \in \mathbb{H} : \text{dist}_{\text{hyp}}(z, A) \leq 1\}.$$

Replacing the radius one by any other number only affects the constants, of course. The area of $N(A)$ is easily seen to be comparable to a number of other geometrically defined quantities, such as the area of all Whitney squares of \mathbb{H} that intersect A , or the area under the minimal Lipschitz function of norm 1 that lies above A . See Figure 1.

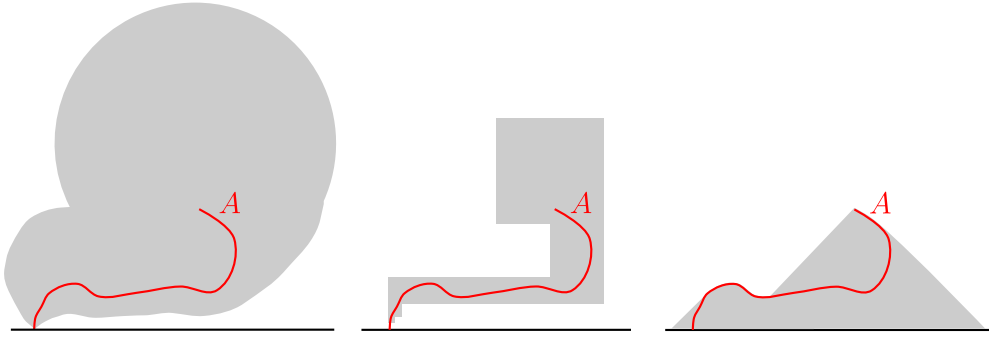


Figure 1: The figure shows the hyperbolic neighborhood of A , the union of all Whitney squares that intersect A , and the area under the minimal Lipschitz function of norm 1 that lies above A . These areas are comparable to the others with universal constants.

Another important quantity is the *conformal radius* of a simply connected domain D with respect to a point $z \in D$, defined as

$$\text{crad}(D, z) := |f'(0)|,$$

where $f: \mathbb{D} \rightarrow D$ is a conformal map from the open unit disk \mathbb{D} onto D with $f(0) = z$. By the Schwarz Lemma and the Koebe One-Quarter Theorem, the conformal radius is comparable to the distance of z to the boundary of D ,

$$\text{dist}(z, \partial D) \leq \text{crad}(D, z) \leq 4 \text{dist}(z, \partial D).$$

This geometric estimate is extremely useful in geometric function theory, but it is useless in the context of the radial Loewner equation, where the domain D is of the form $D = \mathbb{D} \setminus B$ and B is small, so that $\text{crad}(D, 0)$ is close to one. In this situation,

$$\text{dcap}(B) := -\log \text{crad}(\mathbb{D} \setminus B, 0) \asymp 1 - \text{crad}(\mathbb{D} \setminus B, 0)$$

and we have the following theorem.

Theorem 1.2. *If $B \subset \{z \in \mathbb{D} : 1/2 < |z| < 1\}$ and if $\mathbb{D} \setminus B$ is simply connected, then*

$$\text{dcap}(B) \asymp |N(B)|.$$

Theorems 1.1 and 1.2 are essentially equivalent. A special case of Theorem 1.2, Proposition 2.1 below, was proven in [3]. Until recently, Theorem 1.1 did not exist in the published literature, and only existed in form of an unpublished manuscript [4]. Recently, a probabilistic proof of Theorem 1.1 was given in [2]. We give a complex analytic proof of Theorem 1.2 based on [3]. The connection between Theorems 1.1 and 1.2 is then provided by the following simple estimates, valid for all hulls $A \subset \overline{\mathbb{H}}$, all bounded sets $S \subset \mathbb{H}$, and the conformal maps $T_y(z) = \frac{z-iy}{z+iy}$ between \mathbb{H} and \mathbb{D} with $y > 0$:

$$\text{dcap } T_y(A) \sim \frac{2}{y^2} \text{hcap}(A) \quad \text{and} \quad |T_y(S)| \sim \frac{4}{y^2} |S|$$

as $y \rightarrow \infty$, where $a \sim b$ means $\frac{a}{b} \rightarrow 1$.

Remark. After a first version of this paper, we became aware of the papers [5] and [6]. It showed [5] that the half-plane capacity of a set is non-increasing under various notions of symmetrizations. Even though the problems and results of [5] and our work are very different, both exploit the close relation between half-plane capacity and conformal radius. In particular, [5, Lemma 2] is equivalent to Corollary 2.4 below. This allows us to define $\text{hcap}(A)$ as a coefficient in the asymptotic expansion

$$\frac{\text{crad}(\mathbb{H} \setminus A, iy)}{\text{crad}(\mathbb{H}, iy)} = 1 - \frac{2 \text{hcap}(A)}{y^2} + o(y^{-2})$$

as $y \rightarrow \infty$. Under mild assumptions on a (not necessarily simply connected) domain $D \subset \widehat{\mathbb{C}}$ with an accessible boundary point $z_0 \in \partial D$, Dubinin and Vuorinen [6] used a similar asymptotic expansion to define the *relative capacity* of a relatively closed subset $E \subset D$. When $(D, z_0) = (\mathbb{H}, \infty)$, relative capacity coincides with the half-plane capacity [6, Theorem 2.6]. The behavior of the relative capacity under various symmetrizations and under some geometric transformations are also proved in [6], as well as the relation between relative capacity and Schwarzian derivative.

2 Proofs

Let $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$. Denote again $N(B)$ the hyperbolic (with respect to \mathbb{D}) neighborhood of radius one. Let $\widehat{N}(B)$ be the union of $N(B)$ and all of its complementary components with respect to \mathbb{D} that do not contain zero (so that $\mathbb{D} \setminus \widehat{N}(B)$ is simply connected).

For a dyadic interval $J = [\frac{k-1}{2^n}, \frac{k}{2^n}]$ ($n = 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$), consider the dyadic “square”

$$Q_J = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in e^{2\pi i J}, 1 - |z| \leq \frac{1}{2^n} \right\}$$

and its top half $T(Q_J) = \{z \in Q_J : 1 - |z| > \frac{1}{2^{n+1}}\}$. Denote $Q(B)$ the union of all dyadic squares Q with $T(Q) \cap B \neq \emptyset$, see Figure 2. Clearly, $|Q(B)|$, $|N(B)|$ and $|\widehat{N}(B)|$ are comparable with universal constants. The proof of Proposition 2 in [3] showed the following.

Proposition 2.1. *If $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ and $\mathbb{D} \setminus B$ is a simply connected region, then*

$$C_1 |B| \leq \text{dcap}(B) \leq \text{dcap } Q(B) \leq C_2 |Q(B)|,$$

with absolute constants $C_1, C_2 > 0$.

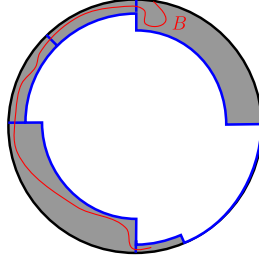


Figure 2: $Q(B)$ is the union of all dyadic squares Q with $T(Q) \cap B \neq \emptyset$.

Sketch of proof. The first inequality in Proposition 2.1 follows from Parseval’s formula. The second inequality is trivial (Schwarz’s Lemma). The final inequality is proved by an inductive procedure as follows. Write

$$Q(B) = \bigcup_{j=1}^{\infty} Q_j,$$

where $\{Q_j\}$ is a disjoint (modulo boundary) family of dyadic squares, arranged so that $|Q_1| \geq |Q_2| \geq \dots$. It suffices to show that

$$(1) \quad \text{dcap} \left(\bigcup_{j=m}^{\infty} Q_j \right) - \text{dcap} \left(\bigcup_{j=m+1}^{\infty} Q_j \right) \asymp |Q_m|$$

for every m . Let f_m be the conformal map from \mathbb{D} onto $\mathbb{D} \setminus \bigcup_{j=m+1}^{\infty} Q_j$, normalized so that $f_m(0) = 0$. Let $K_m = f_m^{-1}(Q_m)$. Inequality (1) is equivalent to $\text{dcap}(K_m) \asymp |Q_m|$. In [3], this last inequality was proved by noting that $\{K_m\}$ is a family of uniform quasi-circles. This allows us to construct two concentric circular hulls of comparable sizes, so that one contains K_m and the other one is contained in K_m . See Figure 3. \square

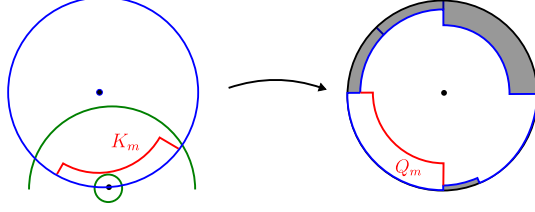


Figure 3: A proof of (1) uses the fact that $\{K_m\}$ is a family of uniform quasi-circles.

The proof of Theorem 1.2 follows from Proposition 2.1, the comparability of $|Q(B)|$ and $|N(B)|$, and the fact that “fattening” the set B does not change $\text{dcap}(B)$ by more than a factor:

Proposition 2.2. *If $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ and $\mathbb{D} \setminus B$ is a simply connected region, then*

$$\text{dcap } \widehat{N}(B) \leq C \text{dcap}(B),$$

where $C > 0$ is an absolute constant.

Proof. Let $\varepsilon > 0$ be a small number that will be chosen later. Define $\widehat{N}_\varepsilon(B)$ the same way as $\widehat{N}(B)$ using hyperbolic radius ε instead of one. We only need to show

$$(2) \quad \text{dcap } \widehat{N}_\varepsilon(B) \lesssim \text{dcap}(B),$$

where $a \lesssim b$ means $a \leq Cb$ for some constant $C > 0$. Once we have proved (2), iterating the inequality $\frac{1}{\varepsilon}$ times gives $\text{dcap } \widehat{N}(B) \lesssim \text{dcap}(B)$. The reverse inequality is trivial by the Schwarz’s Lemma.

Let $\widehat{B} = \widehat{N}_\varepsilon(B)$. Denote $f: \mathbb{D} \rightarrow \mathbb{D} \setminus B$ the conformal map under the normalization $f(0) = 0$ and $f'(0) > 0$. Since $z \mapsto \log \left| \frac{f(z)}{z} \right|$ is harmonic on \mathbb{D} , the mean value property gives

$$(3) \quad \log |f'(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| dt.$$

(If f does not continuously extend to $\partial\mathbb{D}$, either approximate $\mathbb{D} \setminus B$ by smooth regions, or interpret $f(e^{it})$ as an angular limit.) We decompose \mathbb{D} into an union of dyadic layers. For each $n = 1, 2, \dots$, let

$$D_n = \left\{ z \in \mathbb{D} : \frac{1}{2^{n+1}} \leq 1 - |z| < \frac{1}{2^n} \right\} \quad \text{and} \quad B_n = B \cap D_n.$$

From (3) and the elementary fact that $1 - u \asymp -\log(u)$ for $\frac{1}{2} \leq u < 1$, we have

$$(4) \quad \text{dcap}(B) \asymp \sum_{n=1}^{\infty} \frac{\omega_n(0)}{2^n},$$

where $\omega_n(z) = \omega(z, B_n, \mathbb{D} \setminus B)$ denotes the harmonic measure of B_n with respect to the region $\mathbb{D} \setminus B$ and the point z . Define $\widehat{\omega}_n(z)$ similarly using \widehat{B} instead of B . We have

$$(5) \quad \text{dcap}(\widehat{B}) \asymp \sum_{n=1}^{\infty} \frac{\widehat{\omega}_n(0)}{2^n}.$$

Recall that our $\widehat{B} = \widehat{N}_\varepsilon(B)$ depends on ε . Choose $\varepsilon > 0$ so that for each $n \in \mathbb{N}$, every hyperbolic ball (in \mathbb{D}) centered in D_n of radius 2ε is contained in $D_{n-1} \cup D_n \cup D_{n+1}$. This is possible because the hyperbolic distance $\text{dist}_{\text{hyp}}(D_n, D_{n+2}) \asymp 1$. We claim that

$$(6) \quad \widehat{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z)$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{D} \setminus \widehat{B}$. (The inequality $\widehat{\omega}_n(z) \lesssim \omega_n(z)$ is not necessarily true, see Figure 2.) If we can show (6), then (4), (5), (6) imply (2) and complete the proof.

Since both sides of (6) are harmonic functions on $\mathbb{D} \setminus \widehat{B}$, by the maximum principle it suffices to show (6) for z in the boundary of this region. If $z \in \partial(\mathbb{D} \setminus \widehat{B})$ and $z \notin D_n$, then $\widehat{\omega}_n(z) = 0$ and (6) is trivial in this case. Suppose $z \in \partial(\mathbb{D} \setminus \widehat{B})$ and $z \in D_n$ from now on. Since $\widehat{\omega}_n(z) \leq 1$, all we need to show is that the harmonic measure

$$(7) \quad \omega(z, B_{n-1} \cup B_n \cup B_{n+1}, \mathbb{D} \setminus B)$$

is bounded away from zero. Our choice of ε guarantees that the hyperbolic ball $B_{\text{hyp}}(z, 2\varepsilon)$ centered at z with radius 2ε is contained in $D_{n-1} \cup D_n \cup D_{n+1}$, so that the set $E := B_{n-1} \cup B_n \cup B_{n+1}$ connects the hyperbolic circles $\partial B_{\text{hyp}}(z, \varepsilon)$ and $\partial B_{\text{hyp}}(z, 2\varepsilon)$. Now, the maximum principle and the Beurling projection theorem provide a lower bound of (7):

$$\omega(z, E, \mathbb{D} \setminus B) \geq \omega(z, E, B_{\text{hyp}}(z, 2\varepsilon)) \geq \omega(0, [r, 1], \mathbb{D})$$

for some absolute constant $r \in (0, 1)$. □

Proof of Theorem 1.2. By Proposition 2.2 and Proposition 2.1,

$$\text{dcap}(B) \gtrsim \text{dcap} \widehat{N}(B) \gtrsim \left| \widehat{N}(B) \right| \geq |N(B)|.$$

On the other hand,

$$\begin{aligned} \text{dcap}(B) &\leq \text{dcap} Q(\widehat{N}(B)) \\ &\lesssim \left| Q(\widehat{N}(B)) \right| \\ &\lesssim |N(B)| \end{aligned}$$

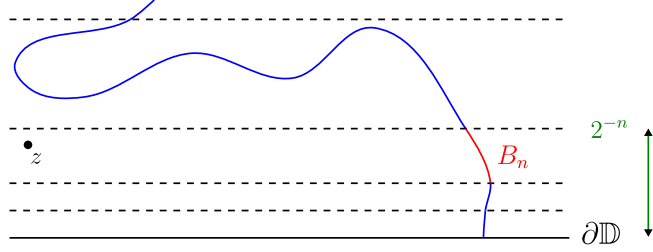


Figure 4: In the proof of Proposition 2.2, it is not generally true that $\tilde{\omega}_n(z) \lesssim \omega_n(z)$. The figure illustrates a situation where $\tilde{\omega}(z) = 1$ but $\omega_n(z)$ is small.

by the Schwarz's Lemma, Proposition 2.1 and the simple facts that $Q(\hat{N}(B)) = Q(N(B))$ and $|Q(N(B))| \lesssim |N(B)|$. \square

We now prove Theorem 1.1 using Theorem 1.2. In fact, the two theorems are equivalent. Proposition 2.3 below gives a connection between half-plane capacity and conformal radius. Roughly speaking, the half-plane capacity of a hull is (up to a constant) asymptotically equal to the change of conformal radius, with respect to a point close to infinity, when the hull is removed. By scaling, we may take the reference point to be i and consider hulls that are small.

Proposition 2.3. *There are absolute constants $C > 0$ and $\varepsilon_0 \in (0, 1)$ such that for any hull $A \subset \mathbb{H}$ with $\sup_{z \in A} |z| \leq \varepsilon \leq \varepsilon_0$,*

$$(8) \quad \left| \frac{2 - \text{crad}(\mathbb{H} \setminus A, i)}{\text{hcap}(A)} - 4 \right| \leq C\varepsilon.$$

For our purpose, we will only need the weaker statement that

$$\lim_{y \rightarrow \infty} \frac{2 - \text{crad}(\mathbb{H} \setminus (y^{-1}A), i)}{\text{hcap}(y^{-1}A)} = 4$$

for every hull $A \subset \mathbb{H}$.

Proof. Note that $\text{crad}(\mathbb{H}, z_0) = 2 \text{Im}(z_0)$. We have

$$\text{crad}(\mathbb{H} \setminus A, i) = \frac{2 \text{Im}g(i)}{|g'(i)|},$$

where $g = g_A: \mathbb{H} \setminus A \rightarrow \mathbb{H}$ is the hydrodynamically normalized conformal map. Write $h = \text{hcap}(A)$ for the sake of notation. We claim that

$$(9) \quad |g(i) - i + ih| \leq C_1 h \varepsilon$$

$$(10) \quad \left| \frac{1}{g'(i)} - 1 + h \right| \leq C_2 h \varepsilon$$

for some absolute constants $C_1, C_2 > 0$. These two inequalities imply

$$\left| \frac{\operatorname{Im} g(i)}{g'(i)} - (1-h)^2 \right| \leq C_3 h \varepsilon,$$

and (8) follows.

To prove (9), we use the Poisson integral representation

$$g(z) - z = \frac{1}{\pi} \int_{-2\varepsilon}^{2\varepsilon} \frac{\operatorname{Im}[g^{-1}(x)]}{g(z) - x} dx \quad (z \in \mathbb{H} \setminus A)$$

which implies that $h = \lim_{z \rightarrow \infty} z[g(z) - z] = \frac{1}{\pi} \int_{-2\varepsilon}^{2\varepsilon} \operatorname{Im}[g^{-1}(x)] dx$ and therefore

$$(11) \quad g(z) - z - \frac{h}{z} = \frac{1}{\pi} \int_{-2\varepsilon}^{2\varepsilon} \operatorname{Im}[g^{-1}(x)] \left(\frac{1}{g(z) - x} - \frac{1}{z} \right) dx \quad (z \in \mathbb{H} \setminus A).$$

On the other hand, we have $|g(z) - z| \leq 3\varepsilon$. (See [1, Corollary 3.44], which showed this estimate using the maximum principle and the observation that it holds for $z \in \partial(\mathbb{H} \setminus A)$.) Using this simple estimate, we can prove that for all $x \in [-2\varepsilon, 2\varepsilon]$,

$$(12) \quad \left| \frac{1}{g(z) - x} - \frac{1}{z} \right| \leq \frac{5\varepsilon}{|z|(|z| - 5\varepsilon)}.$$

Inequality (9) follows from (11) and (12) with $z = i$. Finally, (10) can be proved similarly using

$$\frac{z_2 - z_1}{g(z_2) - g(z_1)} - 1 - \frac{h}{z_1 z_2} = \frac{1}{\pi} \int_{-2\varepsilon}^{2\varepsilon} \operatorname{Im}[g^{-1}(x)] \left(\frac{1}{(g(z_1) - x)(g(z_2) - x)} - \frac{1}{z_1 z_2} \right) dx$$

instead of (11). □

Corollary 2.4. *If $A \subset \mathbb{H}$ is a hull and $B_y = T_y(A) \subset \mathbb{D}$, where $T_y(z) = \frac{z-iy}{z+iy}$ and $y > 0$, then*

$$\lim_{y \rightarrow \infty} \frac{y^2 \operatorname{dcap}(B_y)}{\operatorname{hcap}(A)} = 2.$$

In other words, $\operatorname{dcap}(B_y) \sim \frac{2}{y^2} \operatorname{hcap}(A)$ as $y \rightarrow \infty$.

Proof of Theorem 1.1. Fix a hull $A \subset \mathbb{H}$ and let $T_y(z) = \frac{z-iy}{z+iy}$ as before. When $y > 0$ is large, the hulls $\frac{1}{y}A$ and $B_y = T_y(A) = T_1(y^{-1}A)$ are both small. By Corollary 2.4 and Theorem 1.2,

$$\frac{1}{y^2} \operatorname{hcap}(A) \asymp \operatorname{dcap}(B_y) \asymp |N(B_y)| \asymp |N(y^{-1}A)| \asymp \frac{1}{y^2} |N(A)|.$$

□

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